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# THE STRUCTURE OF INJECTIVE COVERS OF SPECIAL MODULES

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ABSTRACT

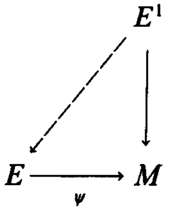
In this paper, we prove that the injective cover of the $R$-module $E(R/\mathfrak{P})/R/\mathfrak{P}$ for a prime ideal $\mathfrak{P}$ of $R$ is the direct sum of copies of $E(R/\mathfrak{Q})$ for prime ideals $\mathfrak{Q} \supset \mathfrak{P}$, and if $\mathfrak{P}$ is maximal, the injective cover is a finite sum of copies of $E(R/\mathfrak{P})$. For a finitely generated $R$-module $M$ with $n$ generators and $G$ an injective $R$-module, we argue that the natural map $G^n \to G^n/\mathrm{Hom}_R(M, G)$ is an injective precover if $\mathrm{Ext}^1_R(M, R) = 0$, and that the converse holds if $G$ is an injective cogenerator of $R$. Consequently, for a maximal ideal $\mathfrak{M}$ of $R$, $\mathrm{depth}_{\mathfrak{M}} R \geqq 2$ if and only if the natural map $E(R/\mathfrak{M}) \to E(R/\mathfrak{M})/R/\mathfrak{M}$ is an injective cover and $\mathrm{depth}_{\mathfrak{M}} R > 0$.

## 1. Introduction

$R$ will denote a commutative noetherian ring.

An injective cover of an $R$-module $M$ is a linear map $\psi : E \to M$ with $E$ an injective $R$-module such that

(1) for any injective $R$-module $E^1$ and linear map $E^1 \to M$, the diagram




can be completed to a commutative diagram

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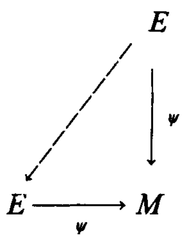
(2) the diagram




can only be completed by automorphisms of $E$.

Hence if an injective cover exists, it is unique up to isomorphism. If $\psi : E \to M$ satisfies (1) and perhaps not (2), it is called an injective precover. We will sometimes simply say $E$ is an injective cover (or precover).

It was shown by Enochs [2] that a ring is noetherian if and only if every $R$-module has an injective precover, or equivalently, every $R$-module has an injective cover. However, examples of injective covers are hard to come by. The first non-trivial example was computed by the first two authors when $R = k[x_1, \dots, x_n]$, $n \geqq 2$ where $k$ is a field. In this case, let $\mathfrak{P} = (x_1, \dots, x_n)$, $R/\mathfrak{P} = k$ (with $x_i k = 0$ for $i = 1, 2, \dots, n$) and $E(k)$ denote the injective envelope of $k$. Then the natural map $E(k) \to E(k)/k$ is an injective cover. This used Northcott's description [6] of $E(k)$ as the inverse polynomial ring $k[x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}]$. The other example is when $R$ is an $n$-dimensional regular local ring with residue field $k$. If $n \geqq 2$, then again the natural map $E(k) \to E(k)/k$ is an injective cover. (See Jenda [4, Corollary 4.2]).

The object of this paper is to generalize these examples and to investigate the injective covers of the $R$-modules $E(R/\mathfrak{P})/R/\mathfrak{P}$ where $\mathfrak{P} \in \operatorname{Spec} R$.

## 2. Main Theorem

THEOREM 2.1. *If $\mathfrak{P}$ is a prime ideal in $R$, then the injective cover of $E(R/\mathfrak{P})/R/\mathfrak{P}$ is the direct sum of copies of $E(R/\mathfrak{Q})$ for prime ideals $\mathfrak{Q}$ such that $\mathfrak{Q} \supset \mathfrak{P}$. If $\mathfrak{P} \notin \operatorname{Ass}(R)$, then the injective cover is the sum of copies of $E(R/\mathfrak{P})$. If $\mathfrak{P}$ is maximal, then the cover is a direct sum of finitely many copies of $E(R/\mathfrak{P})$.*

PROOF. Let $\mathfrak{P}, \mathfrak{Q} \in \operatorname{Spec} R$. If $\mathfrak{P} \not\subset \mathfrak{Q}$, let $r \in \mathfrak{P}$, $r \notin \mathfrak{Q}$. Then multiplication by $r$ on $E(R/\mathfrak{Q})$ is an isomorphism and is zero on $R/\mathfrak{P}$. So $\operatorname{Ext}^1(E(R/\mathfrak{Q}), R/\mathfrak{P}) = 0$. This means that the diagram

$$\begin{array}{ccc} & & E(R/\mathfrak{Q}) \\ & \swarrow & \downarrow \\ E(R/\mathfrak{P}) & \longrightarrow & E(R/\mathfrak{P})/R/\mathfrak{P} \end{array}$$

can be completed when the horizontal map is the natural map and the vertical one is arbitrary.

Hence to construct an injective precover, it suffices to find an injective $R$-module $E$ and a map $E \to E(R/\mathfrak{P})/R/\mathfrak{P}$ such that the diagram

$$\begin{array}{ccc} & & E(R/\mathfrak{Q}) \\ & \swarrow & \downarrow \\ E & \longrightarrow & E(R/\mathfrak{P})/R/\mathfrak{P} \end{array}$$

can be completed whenever $\mathfrak{Q} \supset \mathfrak{P}$ for then

$$\begin{array}{ccc} & & E(R/\mathfrak{Q}) \\ & \swarrow & \downarrow \\ E \oplus E(R/\mathfrak{P}) & \longrightarrow & E(R/\mathfrak{P})/R/\mathfrak{P} \end{array}$$

can be completed for any $\mathfrak{Q} \in \operatorname{Spec} R$. Since every injective $R$-module is the direct sum of copies of such $E(R/\mathfrak{P})$ by Matlis [5], $E \oplus E(R/\mathfrak{P}) \to E(R/\mathfrak{P})/R/\mathfrak{P}$ is a precover. So we let $E$ be the direct sum of sufficiently many copies of the $R$-modules $E(R/\mathfrak{Q})$ where $\mathfrak{Q} \supset \mathfrak{P}$. Then clearly, there is a map $E \to E(R/\mathfrak{P})/R/\mathfrak{P}$ satisfying the above. But an injective cover is a direct summand of a precover, and thus is also a direct sum of such copies.

If $\mathfrak{P} \notin \operatorname{Ass}(R)$, let $K \subset E(R/\mathfrak{P})$ be the field of fractions of $R/\mathfrak{P}$. If $\mathfrak{P} \subsetneq \mathfrak{Q}$ and $E(R/\mathfrak{Q}) \to E(R/\mathfrak{P})/R/\mathfrak{P}$ is a map, consider the composition $E(R/\mathfrak{Q}) \to E(R/\mathfrak{P})/R/\mathfrak{P} \to E(R/\mathfrak{P})/K$. But

$$K = (R/\mathfrak{P})_{\mathfrak{P}}, \quad E(R/\mathfrak{P}) = E(R/\mathfrak{P})_{\mathfrak{P}}, \quad E(R/\mathfrak{P})/K = (E(R/\mathfrak{P})/K)_{\mathfrak{P}}, \quad E(R/\mathfrak{Q})_{\mathfrak{P}} = 0.$$

Thus $E(R/\mathfrak{Q}) \to E(R/\mathfrak{P})/K$ is the zero map. Hence the original map $E(R/\mathfrak{Q}) \to E(R/\mathfrak{P})/R/\mathfrak{P}$ maps $E(R/\mathfrak{Q})$ onto $K/(R/\mathfrak{P})$. But $\mathfrak{P} \notin \operatorname{Ass}(R)$. So there exists an $r \in \mathfrak{P}$ which is not a zero divisor. Thus $E(R/\mathfrak{Q})$ is divisible by $r$. But multiplication by $r$ on $K/(R/\mathfrak{P})$ is zero, and so any map $E(R/\mathfrak{Q}) \to K/(R/\mathfrak{P})$ is zero. Consequently, any map $E(R/\mathfrak{Q}) \to E(R/\mathfrak{P})/R/\mathfrak{P}$ is zero for any prime ideal $\mathfrak{Q} \supsetneq \mathfrak{P}$. Hence the injective cover of $E(R/\mathfrak{P})/R/\mathfrak{P}$ is a direct sum of copies of $E(R/\mathfrak{P})$.

If $\mathfrak{P}$ is maximal, then the injective cover consists of copies of $E(R/\mathfrak{P})$ since $\mathfrak{P} \subset \mathfrak{Q}$ implies $\mathfrak{P} = \mathfrak{Q}$. We now show that a finite number of copies suffices. Let $M^{v}$ denote the Matlis dual $\mathrm{Hom}_R(M, E(R/\mathfrak{P}))$ of an $R$-module $M$. Then $E(R/\mathfrak{P})^{v} = \hat{R}_{\mathfrak{P}}$, the completion of $R$ at $\mathfrak{P}$ and $(E(R/\mathfrak{P})/R/\mathfrak{P})^{v}$ is isomorphic to the maximal ideal $m(\hat{R}_{\mathfrak{P}})$ of $\hat{R}_{\mathfrak{P}}$. Hence by duality, to find an injective cover of the desired form, we only need to argue that for some $n \geqq 1$, and some map $m(\hat{R}_{\mathfrak{P}}) \to \hat{R}_{\mathfrak{P}}^{n}$, the diagram

$$\begin{array}{ccc} m(\hat{R}_{\mathfrak{P}}) & \longrightarrow & \hat{R}_{\mathfrak{P}}^{n} \\ & \searrow & \vdots \downarrow \\ & & \hat{R}_{\mathfrak{P}} \end{array}$$

can be completed for any map $m(\hat{R}_{\mathfrak{P}}) \to \hat{R}_{\mathfrak{P}}$. But $\hat{R}_{\mathfrak{P}}$ is noetherian and so $\mathrm{Hom}_{\hat{R}_{\mathfrak{P}}}(m(\hat{R}_{\mathfrak{P}}), \hat{R}_{\mathfrak{P}})$ is a finitely generated $R$-module. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be a set of generators. Then the map

$$m(\hat{R}_{\mathfrak{P}}) \xrightarrow{(\sigma_1, \sigma_2, \ldots, \sigma_n)} \hat{R}_{\mathfrak{P}}^{n}$$

satisfies the requirements. □

We now show that the injective cover of $E(R/\mathfrak{P})/R/\mathfrak{P}$ can contain copies of $E(R/\mathfrak{Q})$ where $\mathfrak{Q} \supsetneqq \mathfrak{P}$.

Example 2.2. Let $\mathbf{Z}_2$ denote the ring of integers localized at 2, $\mathbf{Q}$ the field of rational numbers. Let $R$ be the ring obtained by the $D \oplus M$ construction where $D = \mathbf{Z}_2$ and $M$ is the simple $\mathbf{Z}_2$-module $\mathbf{Z}_2/(2)$. Then $\mathfrak{P} = 0 \oplus M$ and $\mathfrak{P}' = (2) \oplus M$ are the only prime ideals of $R$. It can be argued that

$$E(R/\mathfrak{P}') = \mathbf{Q}/\mathbf{Z}_2 \oplus \mathbf{Z}_2/(2)$$

with the proper scalar multiplication and that there is a non-zero $R$-homomorphism $E(R/\mathfrak{P}') \to K/(R/\mathfrak{P}) = \mathbf{Q}/\mathbf{Z}_2$ where $K \subset E(R/\mathfrak{P})$ is the field of fractions of $R/\mathfrak{P} = \mathbf{Z}_2$. But $\mathrm{Hom}_R(E(R/\mathfrak{P}'), E(R/\mathfrak{P})) = 0$. So the cover of $E(R/\mathfrak{P})/R/\mathfrak{P}$ cannot consist of copies of $E(R/\mathfrak{P})$ alone. Thus $E(R/\mathfrak{P}')$ must appear in the cover of $E(R/\mathfrak{P})/R/\mathfrak{P}$.

## 3. $E(R/\mathfrak{M})$ as an injective cover

We start with the following:

Theorem 3.1. *Let $M$ be a finitely generated $R$-module with $n$ generators, and $G$ be an injective $R$-module. If* $\mathrm{Ext}_R^1(M, R) = 0$, *then the natural* *map* $G^n \to G^n/\mathrm{Hom}_R(M, G)$ *is an injective precover. The converse holds if* $G$ *is an injective cogenerator of* $R$.

PROOF. Let $E$ be an injective $R$-module. Then $\mathrm{Ext}^1(M, R) = 0$ implies $\mathrm{Hom}(\mathrm{Ext}^1(M, R), E) = 0$. But then $\mathrm{Tor}_1(\mathrm{Hom}(R, E), M) = 0$ by Cartan and Eilenberg [1, VI, pp. 120–121]. So $\mathrm{Hom}(\mathrm{Tor}_1(E, M), G) = 0$. Thus $\mathrm{Ext}^1(E, \mathrm{Hom}(M, G)) = 0$ by Cartan and Eilenberg [1, VI, Proposition 5.1]. But $M = R^n/L$ for some submodule $L$ of $R^n$. So $\mathrm{Hom}(M, G)$ is a submodule of $G^n$. Hence the result follows since $\mathrm{Ext}^1(E, \mathrm{Hom}(M, G)) = 0$ for all injective $R$-modules $E$.

For the converse, note that $G^n \to G^n/\mathrm{Hom}(M, G)$ is an injective precover means $\mathrm{Ext}^1(E, \mathrm{Hom}(M, G)) = 0$ for all injectives $E$. If $G$ is an injective cogenerator, then $\mathrm{Ext}^1(E, \mathrm{Hom}(M, G)) = 0$ means $\mathrm{Ext}^1(M, R) = 0$ as above. □

REMARK. We note that the injective cover of $G^n/\mathrm{Hom}(M, G)$ is a direct summand of $G^n$.

THEOREM 3.2. *Let* $\mathfrak{M}$ *be a maximal ideal in* $R$. *Then the following are equivalent.*

(1) $\mathrm{Ext}^1(R/\mathfrak{M}, R) = 0$.

(2) *The natural map* $E(R/\mathfrak{M}) \to E(R/\mathfrak{M})/R/\mathfrak{M}$ *is an injective cover.*

(3) $\mathrm{Hom}_{\hat{R}_\mathfrak{M}}(m(\hat{R}_\mathfrak{M}), \hat{R}_\mathfrak{M})$ *is cyclic.*

PROOF. (1)⇒(2). By Theorem 3.1 above, $\mathrm{Ext}^1(R/\mathfrak{M}, R) = 0$ implies that the natural map $E(R/\mathfrak{M}) \to E(R/\mathfrak{M})/R/\mathfrak{M}$ is an injective precover since $\mathrm{Hom}(R/\mathfrak{M}, E(R/\mathfrak{M})) \cong R/\mathfrak{M}$. But an injective cover is a direct summand of $E(R/\mathfrak{M})$ and $E(R/\mathfrak{M})$ is indecomposable by Matlis [5]. Hence $E(R/\mathfrak{M})$ is the injective cover.

(2)⇒(1). If $\mathfrak{M}' \in m\,\mathrm{Spec}\,R$ and $\mathfrak{M} \neq \mathfrak{M}'$, let $r \in \mathfrak{M}$, $r \notin \mathfrak{M}'$. Then $r$ is an isomorphism on $E(R/\mathfrak{M}')$ and is zero on $R/\mathfrak{M}$. So $\mathrm{Hom}(R/\mathfrak{M}, E(R/\mathfrak{M}')) = 0$. Therefore

$$\mathrm{Ext}^1\left(E, \mathrm{Hom}\left(R/\mathfrak{M}, \bigoplus_{\mathfrak{P} \in m\,\mathrm{Spec}\,R} E(R/\mathfrak{P})\right)\right) = \mathrm{Ext}^1(E, \mathrm{Hom}(R/\mathfrak{M}, E(R/\mathfrak{M})))$$

$$= \mathrm{Ext}^1(E, R/\mathfrak{M}).$$

Hence $\mathrm{Ext}^1(E, R/\mathfrak{M}) = 0$ for all injective $R$-modules $E$ implies $\mathrm{Ext}^1(R/\mathfrak{M}, R) = 0$ by Theorem 3.1 since $\bigoplus_{\mathfrak{P} \in m\,\mathrm{Spec}\,R} E(R/\mathfrak{P})$ is an injective cogenerator of $R$ by Ishikawa [3, Corollary 3.2].

(2)$\Rightarrow$(3). By Matlis duality, if $E(R/\mathfrak{M})^n \rightarrow E(R/\mathfrak{M})/R/\mathfrak{M}$ is an injective cover, then $n$ is the least number of generators of $\mathrm{Hom}_{\hat{R}_{\mathfrak{M}}}(m(\hat{R}_{\mathfrak{M}}), \hat{R}_{\mathfrak{M}})$.

(3)$\Rightarrow$(2). If $\mathrm{Hom}_{\hat{R}_{\mathfrak{M}}}(m(\hat{R}_{\mathfrak{M}}), \hat{R}_{\mathfrak{M}})$ is cyclic, then as in the proof of Theorem 2.1, $E(R/\mathfrak{M}) \rightarrow E(R/\mathfrak{M})/R/\mathfrak{M}$ is an injective precover and so is an injective cover. □

The following corollary generalizes the examples discussed in Section 1.

COROLLARY 3.3. *Let* $\mathfrak{M}$ *be a maximal ideal of* $R$. *Then* $\mathrm{depth}_{\mathfrak{M}} R \geqq 2$ *if and only if the natural map* $E(R/\mathfrak{M}) \rightarrow E(R/\mathfrak{M})/R/\mathfrak{M}$ *is an injective cover and* $\mathrm{depth}_{\mathfrak{M}} R > 0$.

COROLLARY 3.4. *Let* $\mathfrak{M}$ *be a maximal ideal of* $R$. *Then* $\mathrm{depth}_{\mathfrak{M}} R = 1$ *if and only if the injective cover of* $E(R/\mathfrak{M})/R/\mathfrak{M}$ *has at least two copies of* $E(R/\mathfrak{M})$ *and* $\mathrm{depth}_{\mathfrak{M}} R > 0$.

PROOF. If $\mathrm{depth}_{\mathfrak{M}} R = 1$, then the natural map $E(R/\mathfrak{M}) \rightarrow E(R/\mathfrak{M})/R/\mathfrak{M}$ is not an injective cover by Corollary 3.3 above. But the injective cover of $E(R/\mathfrak{M})/R/\mathfrak{M}$ is a sum (finite) of copies of $E(R/\mathfrak{M})$ by Theorem 2.1. Hence the cover has more than one copy of $E(R/\mathfrak{M})$.

The converse follows from Corollary 3.3. □

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